3

COLLIDING IMPULSIVE GRAVITATIONAL WAVES

One of the first exact solutions of Einstein's equations describing the mutual scattering of gravitational waves was that of Khan and Penrose (1971). This describes the collision and subsequent interaction of two plane impulsive gravitational waves and their subsequent interaction. Following this pioneering work, many more exact solutions have been obtained. However, the general character of almost all solutions that have subsequently been found is basically the same as that of the original solution of Khan and Penrose. It is therefore appropriate to devote a chapter, at this early stage in a discussion of the subject, to this particular solution. The aim is to describe it in detail, pointing out some of its properties and the problems that it highlights, to which further attention will be given in subsequent chapters.

3.1 The approaching waves

In the Khan–Penrose solution the approaching waves are plane impulsive gravitational waves. Metrics describing such waves are well known. In a flat background, one of the approaching waves may conveniently be described by the Brinkmann–Peres–Takeno line element

$$\mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}r + \delta(u)(X^2 - Y^2)\mathrm{d}u^2 - \mathrm{d}X^2 - \mathrm{d}Y^2 \tag{3.1}$$

where $\delta(u)$ is the Dirac delta function, here representing the impulsive wave component. Although the metric in this case is distribution valued, it will be shown later that it does in fact give rise to a geometrically acceptable space-time.

The line element (3.1) involves a null coordinate u, and the plane impulsive wave occurs on the null hypersurface $u = 0$. The opposing wave may also initially be described by the same line element, but with the null coordinate u replaced by another null coordinate v. Naturally, another space-like coordinate must replace r, but for 'head on' collisions the other space-like coordinates X and Y may be retained provided that the two approaching waves have their polarization vectors aligned.

It is found to be most convenient to describe the entire space-time using the two null coordinates u and v. The initial wave (3.1) can be expressed in terms of these two coordinates by using the transformation

$$\begin{aligned} u &= u \\ r &= v - \tfrac{1}{2}\Theta(u)(1-u)x^2 + \tfrac{1}{2}\Theta(u)(1+u)y^2 \\ X &= \big(1 - u\Theta(u)\big)x \\ Y &= \big(1 + u\Theta(u)\big)y \end{aligned} \tag{3.2}$$

where $\Theta(u)$ is the Heaviside step function. It may be noted that the discontinuity in this transformation is related to the discontinuity in the line element (3.1). At this stage, this transformation may be considered as a purely symbolic manipulation which will later be shown to be geometrically acceptable. Using the transformation (3.2), the line element (3.1) takes the form

$$\mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}v - \big(1 - u\Theta(u)\big)^2\mathrm{d}x^2 - \big(1 + u\Theta(u)\big)^2\mathrm{d}y^2. \tag{3.3}$$

The gravitational wave is here described by the single component

$$\Psi_4 = \delta(u). \tag{3.4}$$

Strictly, the line element (3.3) with wave component (3.4) only describes one wave as it approaches the other. It therefore only applies to the region $v < 0$. The opposing wave, prior to the collision, is described by the same line element but with u and v interchanged. It is thus described by the component

$$\Psi_0 = \delta(v). \tag{3.5}$$

Since this also only describes the approaching wave, it similarly is defined only in the region $u < 0$. The initial situation of two approaching waves has now been defined. The question to be discussed is what happens to the two waves when they meet and pass through each other. How do they interact gravitationally?

At this point it is convenient to divide space-time up into four distinct regions, as illustrated in Figure 3.1. Region I, where $u < 0$ and $v < 0$, is the flat background having line element

$$\mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}v - \mathrm{d}x^2 - \mathrm{d}y^2. \tag{3.6}$$

Region II, where $u \geq 0$ and $v < 0$, contains the wave with line element (3.3)

$$\mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}v - (1-u)^2\mathrm{d}x^2 - (1+u)^2\mathrm{d}y^2. \tag{3.7}$$

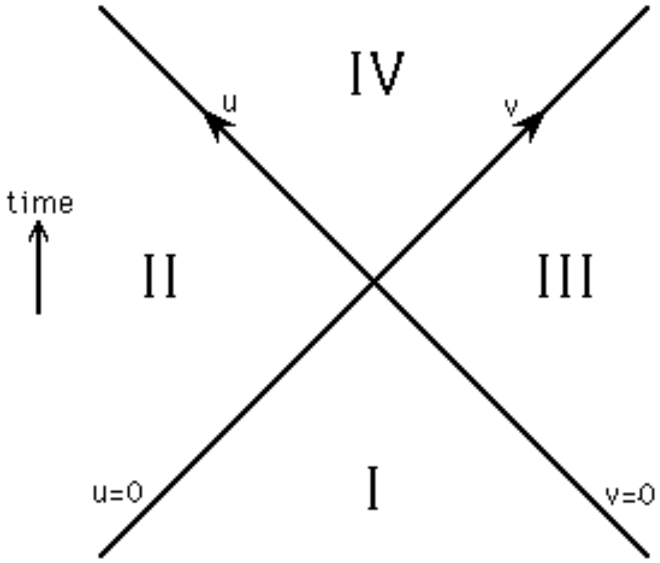

Figure 3.1 Space-time is conveniently divided into four regions as shown. Two space-like coordinates have been suppressed. Region I is the background, regions II and III contain the approaching waves, and region IV is the interaction region following the collision at the point $u = 0$, $v = 0$.

Region III, where $v \geq 0$ and $u < 0$, contains the opposing wave, which is described by the line element

$$\mathrm{d}s^2 = 2\mathrm{d}u\mathrm{d}v - (1 - v)^2\mathrm{d}x^2 - (1 + v)^2\mathrm{d}y^2. \tag{3.8}$$

Region IV is the interaction region $u \geq 0$ and $v \geq 0$. Its line element will have to be found as a solution of Einstein's equations, subject to appropriate boundary conditions set on the two null surfaces $v = 0$, $u \geq 0$ and $u = 0$, $v \geq 0$. In this case initial data is well set, so a unique solution exists in this region.

Before presenting the solution for region IV, it is worth noticing that the metrics (3.7) and (3.8) for regions II and III are singular on the null hypersurfaces $u = 1$ and $v = 1$ respectively. Such singularities do not occur in the line elements of the form (3.1). It is therefore tempting to regard them simply as coordinate singularities, since they can be removed by transformations of the type (3.2). In fact these singularities do seem to have a particular significance, and they will be discussed in detail in the following sections.

3.2 The solution describing the interaction

The field equations for the interaction region IV, and the appropriate boundary conditions will be discussed in full in the following chapters.

At this point it is only appropriate to state the resulting line element as given by Khan and Penrose (1971). This has the form

$$\begin{aligned} \mathrm{d}s^2 = {} & 2\frac{(1-u^2-v^2)^{3/2}}{\sqrt{1-u^2}\sqrt{1-v^2}(uv+\sqrt{1-u^2}\sqrt{1-v^2})^2}\mathrm{d}u\mathrm{d}v \\ & -(1-u^2-v^2)\left(\frac{(1-u\sqrt{1-v^2}-v\sqrt{1-u^2})}{(1+u\sqrt{1-v^2}+v\sqrt{1-u^2})}\mathrm{d}x^2\right. \\ & \left. +\frac{(1+u\sqrt{1-v^2}+v\sqrt{1-u^2})}{(1-u\sqrt{1-v^2}-v\sqrt{1-u^2})}\mathrm{d}y^2\right). \end{aligned} \tag{3.9}$$

Using a notation that will be defined later, the gravitational field can be described by the components

$$\begin{aligned} \Psi_0^\circ &= \frac{1}{\sqrt{1-u^2}}\delta(v) + \frac{3u\sqrt{1-u^2}(uv+\sqrt{1-u^2}\sqrt{1-v^2})}{(1-v^2)(1-u^2-v^2)^2} \\ \Psi_2^\circ &= \frac{(uv+\sqrt{1-u^2}\sqrt{1-v^2})^2}{\sqrt{1-u^2}\sqrt{1-v^2}(1-u^2-v^2)^2} - \frac{uv}{(1-u^2-v^2)^2} \\ \Psi_4^\circ &= \frac{1}{\sqrt{1-v^2}}\delta(u) + \frac{3v\sqrt{1-v^2}(uv+\sqrt{1-u^2}\sqrt{1-v^2})}{(1-u^2)(1-u^2-v^2)^2}. \end{aligned} \tag{3.10}$$

It may be noticed that the impulsive wave components continue after the point of collision at $u=0$, $v=0$, but with a scaled magnitude. Both wave components also develop tails, so that the components Ψ_0 and Ψ_4 are non-zero throughout the interaction region. The two waves may therefore be considered to scatter each other. Another most significant feature that is demonstrated by the components (3.10) is that the component Ψ_2 also becomes non-zero. This is in fact a familiar feature of interacting waves. In fact it will be shown later that it is not possible for a vacuum space-time to contain only the components Ψ_0 and Ψ_4.

It should also be noticed that all the components described by (3.10) become unbounded on the hypersurface $u^2+v^2=1$ on which the line element (3.9) is singular. That this corresponds to a scalar polynomial curvature singularity can be seen by considering the scalar invariant

$$3{\Psi_2}^2 + \Psi_0\Psi_4 = \delta(u)\delta(v) + \frac{3S^4(S^4+S^2P+P^2)}{(1-u^2-v^2)^7} \tag{3.11}$$

where $S = uv+\sqrt{1-u^2}\sqrt{1-v^2}$, and $P = uv\sqrt{1-u^2}\sqrt{1-v^2}$.

3.3 The structure of the solution

The most important feature illustrated by this solution is the existence of the future space-like curvature singularity at $u^2 + v^2 = 1$. It may be noticed that this is connected at the boundaries to the coordinate singularities in regions II and III. If these coordinate singularities are considered as having no physical significance, then the naive structure of the solution is as described in Figure 3.2.

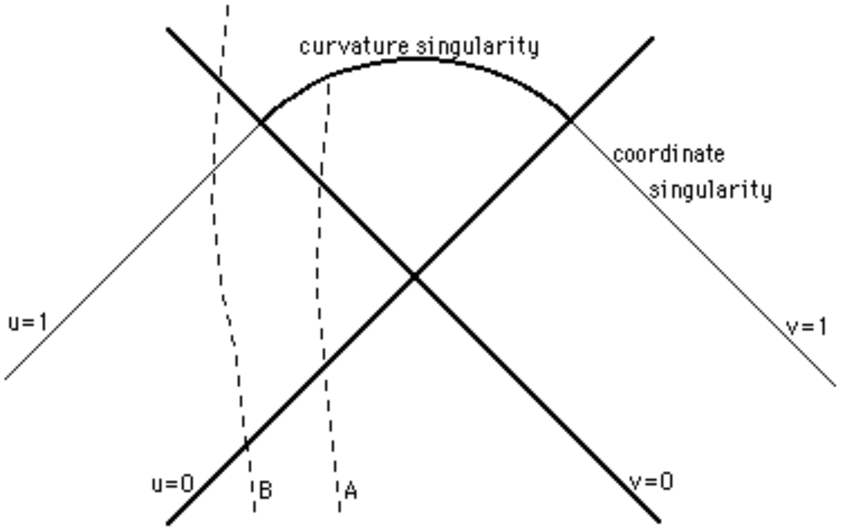

Figure 3.2 The apparent singularity structure of the Khan–Penrose solution. There is a curvature singularity in region IV which is attached to coordinate singularities in regions II and III. Apparent worldlines of particles A and B are referred to in the text.

Refering to this figure, we may consider the worldline of a test particle A, which first encounters one gravitational wave, and then the other, and must subsequently fall into a curvature singularity after a finite proper time. This would seem to be the fate of all objects that detect two plane gravitational waves. Such situations are not unfamiliar in general relativity. They occur for example in cosmology if the cosmic expansion ceases. The present situation may be similar, in that once a general contraction is established, a future singularity becomes inevitable. The possibility of such an interpretation is at least a little comforting, in that it suggests that the singularity may be removed to a cosmological time scale in the future. This will need to be considered. We will clearly also have to spend some time trying to establish whether or not such singularities are a generic feature of colliding waves. In fact it will be shown in subsequent chapters that curvature singularities of this type are a general feature of colliding wave solutions whenever the initial data has global plane symmetry, and the background in region I is flat. However, there is a large class of notable exceptions in which this curvature singularity is replaced by a Cauchy horizon.

It may be pointed out at this stage, however, that the time between the collision and the subsequent singularity is inversely proportional to the strength of the waves. This can be seen simply by replacing u by au and v by bv in the above solution. In this case, the two approaching waves prior to the collision are given by $\Psi_4 = a\delta(u)$ and $\Psi_0 = b\delta(v)$. The singularity then occurs on the space-like surface given by $1 - a^2u^2 - b^2v^2 = 0$.

Refering again to Figure 3.2, consider the worldline of the test particle B. This encounters a gravitational wave and, after some time, appears to encounter the coordinate singularity in region II. If we consider this to be merely a coordinate singularity, then the particle may simply be able to pass through it. It would then subsequently meet the opposing wave and, having passed through this, it would then continue indefinitely into the future. However, once the particle has passed the second gravitational wave, it will be able to look back and see a naked curvature singularity. Such possibilities must at least be considered with suspicion in view of the cosmic censorship hypothesis suggested by Penrose. It will therefore be necessary to analyse in detail the character of the singularities in regions II and III. In fact it will be shown that paths of the type described are not permitted. It will be argued that pictures such as Figures 3.1 and 3.2 can be misleading in that they only illustrate the time-like hypersurface $x = y = 0$, and do not indicate the causal structure of the complete space-time.